NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 4275

DYNAMIC STABILITY OF VEHICLES TRAVERSING ASCENDING OR DESCENDING PATHS THROUGH THE ATMOSPHERE

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SUMMARY

An analysis is given of the oscillatory motions of vehicles which traverse ascending and descending paths through the atmosphere at high speed. The specific case of a skip path is examined in detail, and this leads to a form of solution for the oscillatory motion which should recur over any trajectory. The distinguishing feature of this form is the appearance of the Bessel rather than the trigonometric function as the characteristic mode of oscillation.

INTRODUCTION

The speed of many rocketcraft and some aircraft is so great that they possess ample kinetic energy to permit ejection from the atmosphere. With the satellite vehicle, the ballistic rocket, and the skip rocket, such trajectories are, of course, intentional. With high-speed aircraft and with glide rockets, although sojourns into space may purposely be made, they may also accidentally occur if, when in climbing flight, the dynamic pressure falls so rapidly that the vertical momentum cannot be checked to prevent such a course.

When a vehicle is in space it may have arbitrary orientation with respect to the flight path and this orientation may change with time. In the absence of air, the history of this orientation can only be changed by the use of some form of reaction control. However, as the vehicle descends into the atmosphere it will develop motions in response to the aerodynamic forces and moments which come into play. If the vehicle possesses positive static and dynamic stability, these motions will subside into the types with which we have become accustomed from more conventional aircraft. The whole history of the motion is of interest to the designer, both for guidance system and structural design and in determining the feasibility of human habitation.

The dynamic motion of ballistic vehicles has been the subject of several theoretical investigations. Oswald (ref. 1) considered the motions of an unguided rocket in its flight out of the atmosphere. The general problem of a ballistic missile entering the atmosphere was treated by Friedrich and Dore in reference 2. The analysis of reference 3 treated the entry problem with certain simplifying assumptions regarding the trajectory and the atmosphere in order to clarify the nature of the problem. For all the above analyses, the zero-lift trajectory was used.

It is the purpose of this paper to investigate the motion history for the cases for which aerodynamic lift is important in changing the trajectory. In common with the analyses cited above, it will be assumed that the trajectory lies in a vertical plane and that disturbing forces are symmetric with respect to the vehicle's vertical plane of symmetry. This reduces the problem to one of longitudinal motions only. In order to illustrate the mechanics of the problem more clearly, the analysis is concerned first with the specific example of a vehicle which descends through the atmosphere on a skip trajectory, executes its turn, and is ejected again from the atmosphere. A detailed study of this problem reveals the nature of the essential parameters, and this leads to a more general form of solution which can be used to calculate the oscillatory history over any given trajectory.

SYMBOLS

A reference area

C_D drag coefficient, $\frac{\text{drag}}{qA}$

C_L lift coefficient, $\frac{\text{lift}}{qA}$

C_m pitching-moment coefficient, $\frac{\text{pitching moment}}{qAl}$

C_{L_α} rate of change of lift coefficient with angle of attack, $\left(\frac{\partial C_L}{\partial \alpha}\right)_{\alpha \to 0}$

C_{m_α} rate of change of pitching-moment coefficient with angle of attack, $\left(\frac{\partial C_m}{\partial \alpha}\right)_{\alpha \to 0}$

$C_{m_{\dot{\alpha}}}$ rate of change of moment coefficient with time rate of change of angle-of-attack parameter $\frac{\dot{\alpha} l}{V}$, $\left(\frac{\partial C_m}{\partial \dot{\alpha} l/V}\right)_{\dot{\alpha} \to 0}$

C_{m_q} rate of change of moment coefficient with pitching velocity parameter $\frac{\dot{\theta} l}{V}$, $\left(\frac{\partial C_m}{\partial \dot{\theta} l/V}\right)_{\dot{\theta} \to 0}$

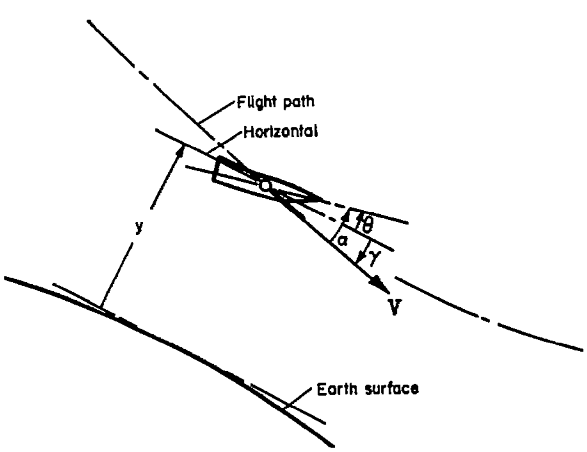
C_{m_δ}	rate of change of moment coefficient with control deflection, $\left(\frac{\partial C_m}{\partial \delta}\right)_{\delta \to 0}$
e	Naperian base
g	acceleration due to gravity
I	pitching moment of inertia about center of gravity
$\left.\begin{matrix} J_0(\ ) \\ J_1(\ ) \end{matrix}\right\}$	Bessel functions of first kind of zero and first order
K_1	dynamic stability parameter (eq. (15))
K_2	static stability parameter (eq. (21))
$\frac{L}{D}$	lift-drag ratio
l	body length and reference length for moment coefficient evaluation
m	vehicle mass
q	dynamic pressure, $\frac{1}{2} \rho V^2$
R	radius of earth
s	distance measured along path of static trajectory
t	time
V	flight speed
V_E	flight speed on entrance to the atmosphere
W	vehicle weight
x	distance measured along earth
y	altitude
$\left.\begin{matrix} Y_0(\ ) \\ Y_1(\ ) \end{matrix}\right\}$	Bessel functions of second kind of zero and first order
α	angle of attack (sketch (a))
α_E	initial value of oscillatory angle of attack on entrance to the atmosphere
β	density parameter (eq. (8))

γ	flight path angle, $\alpha - \theta$ (sketch (a))
γ_E	static flight path angle on entrance to the atmosphere
γ_s	static flight path angle
δ	control deflection angle
θ	angle of pitch (sketch (a))
ξ	dummy variable
ρ	air density
ρ_o	air density at sea level
σ	radius of gyration
ω	oscillation circular frequency, radians/sec

ANALYSIS

The plan of the analysis to be undertaken is as follows: We shall present first the equations which apply generally to the longitudinal motions of a vehicle moving through the atmosphere at high speed. Then, several sections will be devoted to a study of the equations as they apply specifically to the skip trajectory, as this will reveal a number of important features and permissible simplifications. In a final section, these features and simplifications will be exploited to derive a general form of solution for the oscillatory motion which is applicable over any given trajectory.

Equations of Motion



Sketch (a)

Consider a vehicle moving through the atmosphere at high speed, which at time t is approaching the earth along a flight path depressed from the horizontal by the positive angle γ (see sketch (a)). The full set of equations of motion defining the vehicle's path and the oscillations of the vehicle about that path may be written as

$$\left.\begin{aligned} -m\dot{V} - C_D qA + mg \sin \gamma &= 0 \\ mV\dot{\gamma} + C_L qA + m\left(\frac{V^2}{R+y} - g\right)\cos \gamma &= 0 \\ I\ddot{\theta} - qAl \sum C_m &= 0 \end{aligned}\right\} \tag{1}$$

wherein

$$\theta = \alpha - \gamma$$

$$C_L = C_{L_\alpha}\alpha$$

$$\sum C_m = C_{m_\alpha}\alpha + C_{m_q}\frac{\dot{\theta}l}{V} + C_{m_{\dot{\alpha}}}\frac{\dot{\alpha}l}{V} + C_{m_\delta}\delta$$

and a dot over a symbol denotes differentiation with respect to time. Because of the dependence on time of density ρ and velocity V, equations (1) are nonlinear; their solution cannot be effected analytically unless simplifying assumptions are made. Herein we assume the following:

1. The aerodynamic coefficients in equations (1) are essentially independent of Mach number, a justifiable assumption for the high flight velocities of interest here.

2. The drag coefficient C_D is independent of angle of attack and pitching velocity, a justifiable assumption so long as these variables remain small.

3. The equations may be broken down into two sets of equations, one set defining the "static trajectory" of the vehicle's center of gravity, the other describing the oscillations of the vehicle about its center of gravity as well as the oscillatory motions of the center of gravity about the static trajectory. The justification for this procedure will be evident from the following considerations. Each of the dependent variables α, γ, and θ in equations (1) is comprised of a uniform variation with time plus an oscillatory variation. Hence, let us make the following substitutions:

$$\left.\begin{aligned} \gamma(t) &= \gamma_s(t) + \gamma_o(t) \\ \alpha(t) &= \alpha_s(t) + \alpha_o(t) \\ \theta(t) &= \theta_s(t) + \theta_o(t) \end{aligned}\right\} \tag{2}$$

wherein the subscript s denotes the uniform part of each motion and o the oscillatory part. Now, under the restriction that γ_o be a small quantity, so that

$$\frac{\gamma_o}{\gamma_s} \ll 1$$

$$\sin \gamma_s\left(1 + \frac{\gamma_o}{\gamma_s}\right) \approx \sin \gamma_s$$

$$\cos \gamma_s\left(1 + \frac{\gamma_o}{\gamma_s}\right) \approx \cos \gamma_s$$

equations (1) may be grouped as follows:

$$\left.\begin{aligned}
-m\dot{V} - C_D qA + mg \sin \gamma_s &= 0 \\
\left[mV\dot{\gamma}_s + qAC_{L_\alpha}\alpha_s + m\left(\frac{V^2}{R+y} - g\right)\cos \gamma_s\right] + \left[mV\dot{\gamma}_o + qAC_{L_\alpha}\alpha_o\right] &= 0 \\
\left[I\ddot{\theta}_s - qAl\left(C_{m_\alpha}\alpha_s + C_{m_q}\frac{\dot{\theta}_s l}{V} + C_{m_{\dot{\alpha}}}\frac{\dot{\alpha}_s l}{V} + C_{m_\delta}\delta\right)\right] + & \\
\left[I\ddot{\theta}_o - qAl\left(C_{m_\alpha}\alpha_o + C_{m_q}\frac{\dot{\theta}_o l}{V} + C_{m_{\dot{\alpha}}}\frac{\dot{\alpha}_o l}{V}\right)\right] &= 0
\end{aligned}\right\} \quad (3)$$

In each of the last two equations, the second bracket is a sum of oscillatory quantities whereas the first bracket is not; under the assumption that over a given cycle of oscillation the velocity V and dynamic pressure q remain essentially constant, the average of the second group of terms over the cycle must be near zero. Hence, so must the average of the first. But since the behavior of the first group of terms is not oscillatory, in order that its average be near zero its value at any time must be near zero. Therefore, we can assume that each group of terms in equations (3) is independently zero. We have then for the nonoscillatory or "static" trajectory

$$\left.\begin{aligned}
-m\dot{V} - C_D qA + mg \sin \gamma_s &= 0 \\
mV\dot{\gamma}_s + qAC_{L_\alpha}\alpha_s + m\left(\frac{V^2}{R+y} - g\right)\cos \gamma_s &= 0 \\
I\ddot{\theta}_s - qAl\left(C_{m_\alpha}\alpha_s + C_{m_q}\frac{\dot{\theta}_s l}{V} + C_{m_{\dot{\alpha}}}\frac{\dot{\alpha}_s l}{V} + C_{m_\delta}\delta\right) &= 0 \\
\theta_s = \alpha_s - \gamma_s &
\end{aligned}\right\} \quad (4)$$

and for the oscillatory motion

$$\left.\begin{aligned} mV\dot{\gamma}_o + qAC_{L_\alpha}\alpha_o &= 0 \\ I\ddot{\theta}_o - qAl\left(C_{m_\alpha}\alpha_o + C_{m_q}\frac{\dot{\theta}_o l}{V} + C_{m_{\dot{\alpha}}}\frac{\dot{\alpha}_o l}{V}\right) &= 0 \\ \theta_o = \alpha_o - \gamma_o \end{aligned}\right\} \tag{5}$$

The set of equations (5) can be combined into a single equation for the oscillatory angle of attack of the form

$$\ddot{\alpha}_o + f_1(t)\dot{\alpha}_o + f_2(t)\alpha_o = 0 \tag{6}$$

wherein

$$\left.\begin{aligned} f_1(t) &= C_{L_\alpha}\frac{\rho VA}{2m} - \left(C_{m_q} + C_{m_{\dot{\alpha}}}\right)\frac{\rho VAl^2}{2I} \\ f_2(t) &= \frac{d}{dt}\left(C_{L_\alpha}\frac{\rho VA}{2m}\right) - \frac{C_{m_q}C_{L_\alpha}}{Im}\left(\frac{\rho VAl}{2}\right)^2 - C_{m_\alpha}\frac{\rho V^2Al}{2I} \end{aligned}\right\} \tag{7}$$

Skip Trajectory

The differential equations (4) to (6) are applicable to the analysis of arbitrary static flight paths and their accompanying oscillatory histories. We now consider the application of equations (4) to (6) to the specific case of a skip trajectory under the simplifying condition that L/D is constant.

Static trajectory.- Since C_D has been assumed constant, the specification that L/D be a constant means that α_s is also fixed. This permits the variation of flight path angle with time to be determined from the first two of equations (4). The third equation specifies how the control motion is to be programmed in order to maintain constant L/D. We shall not be concerned with the latter problem here.

The static skip trajectory for constant lift-drag ratio has been determined in reference 4 under the assumption that: (1) the air density varies with altitude in the manner

$$\rho = \rho_o e^{-\beta y} \tag{8}$$

wherein ρ_o and β are constants, and (2) the effect on the path of gravity and the term $V^2/(R + y)$ may be ignored. The first assumption is an adequate one (see ref. 5) over the range of altitudes for which the aerodynamic forces are important in determining the path, provided the constants ρ_o and β are appropriately chosen. In most cases the values

$$\left.\begin{aligned} \rho_o &= 0.0034 \text{ slugs/ft}^3 \\ \beta &= \frac{1}{22{,}000 \text{ ft}} \end{aligned}\right\} \tag{9}$$

are satisfactory and will be used in this report. The second assumption is necessary to keep the functional relations simple. It is a satisfactory one when the flight velocity is high and the aerodynamic forces experienced during the skip are large (see ref. 4). For present purposes it is felt to be sufficient, since our concern is more with the nature of the motion problem rather than with an accurate evaluation of the motion. Nevertheless, the adequacy of this assumption will be reviewed later in this report as it affects the quantitative results.

If one defines γ_s as the instantaneous static flight path angle and γ_E as the static flight path angle at the "entrance" to the atmosphere, then the analysis of reference 4 gives the velocity and density dependence on γ_s as

$$\left.\begin{aligned} V &= V_E e^{\frac{\gamma_s - \gamma_E}{L/D}} \\ \rho &= \rho_o e^{-\beta y} = \frac{2\beta m}{C_L A}(\cos\gamma_s - \cos\gamma_E) \\ C_L &= C_{L_\alpha}\alpha_s = \text{constant} \end{aligned}\right\} \tag{10}$$

The coordinates of the flight path can be found from equations (10) to be

$$\left.\begin{aligned} y &= \frac{1}{\beta} \ln\left[\frac{C_L \rho_o A}{2\beta m(\cos\gamma_s - \cos\gamma_E)}\right] \\ x &= \mp \frac{1}{\beta}\left[\left|\gamma_s\right| + \frac{1}{\tan\gamma_E} \ln\left(\frac{\tan\gamma_E/2 + \tan\left|\gamma_s/2\right|}{\tan\gamma_E/2 - \tan\left|\gamma_s/2\right|}\right)\right] \end{aligned}\right\} \tag{11}$$

In the equation for x the minus sign is used for the incoming path and the plus sign for the outgoing.

The path length s measured from the bottom of the skip can be shown to be

$$s = \mp \frac{1}{\beta \sin\gamma_E} \ln\left(\frac{\tan\gamma_E/2 + \tan\left|\gamma_s/2\right|}{\tan\gamma_E/2 - \tan\left|\gamma_s/2\right|}\right) \tag{12}$$

wherein, as with x, the minus sign is used for the incoming path and the plus sign for the outgoing.

Oscillatory motion.- Having eliminated α_s from equations (10) and γ_o from equations (5), henceforth we shall omit the subscripts s and o as no confusion should ensue. It will be understood that in referring to α we mean the oscillatory angle of attack, whereas in referring to γ we mean the flight path angle of the static trajectory.

Consider now equation (6) for the oscillatory angle of attack. It will be found convenient to consider the independent variable as γ $(=\gamma_s)$ rather than t. Using the relations

$$\left.\begin{aligned}
\frac{d\alpha}{dt} &= \alpha'(\gamma)\frac{d\gamma}{dt} \\
\frac{d^2\alpha}{dt^2} &= \alpha''(\gamma)\left(\frac{d\gamma}{dt}\right)^2 + \alpha'(\gamma)\frac{d^2\gamma}{dt^2} \\
\frac{d\gamma}{dt} &= -\beta V_E e^{\frac{\gamma - \gamma_E}{L/D}}(\cos\gamma - \cos\gamma_E) \\
\frac{d^2\gamma}{dt^2} &= \left(\frac{d\gamma}{dt}\right)^2\left(\frac{1}{L/D} - \frac{\sin\gamma}{\cos\gamma - \cos\gamma_E}\right)
\end{aligned}\right\} \tag{13}$$

we find, after some manipulation, that equation (6) becomes

$$\alpha''(\gamma) + f_3(\gamma)\alpha'(\gamma) + f_4(\gamma)\alpha(\gamma) = 0 \tag{14}$$

where

$$\left.\begin{aligned}
f_3(\gamma) &= 2K_1 - \frac{\sin\gamma}{\cos\gamma - \cos\gamma_E} \\
f_4(\gamma) &= \frac{1}{(\cos\gamma - \cos\gamma_E)}\left[\frac{C_{L_\alpha}}{C_L}\sin\gamma - \frac{C_{m_\alpha}}{C_L}\frac{1}{\beta l}\left(\frac{l}{\sigma}\right)^2\right] - \\
&\quad \left[\frac{C_{L_\alpha}}{C_L}\frac{1}{L/D} + \frac{C_{m_q}C_{L_\alpha}}{C_L^2}\left(\frac{l}{\sigma}\right)^2\right] \\
K_1 &= \frac{1}{2}\left[\frac{1}{L/D} - \frac{C_{L_\alpha}}{C_L} + \frac{C_{m_q} + C_{m_{\dot\alpha}}}{C_L}\left(\frac{l}{\sigma}\right)^2\right]
\end{aligned}\right\} \tag{15}$$

and the primes indicate differentiation with respect to γ. Finally, following the method of Friedrich and Dore (ref. 2), we remove the first-derivative term in equation (14) by means of the transformation

$$\alpha(\gamma) = \bar{\alpha}(\gamma)e^{-\frac{1}{2}\int^{\gamma} f_3(\xi)d\xi} \tag{16}$$

whereupon equation (14) becomes

$$\bar{\alpha}''(\gamma) + M(\gamma)\bar{\alpha}(\gamma) = 0 \tag{17}$$

with

$$M(\gamma) = f_4(\gamma) - \frac{1}{2} f_3'(\gamma) - \frac{1}{4} f_3^2(\gamma) \tag{18}$$

Alternate solutions of equation (14) are presented in the following two sections.

Method of Friedrich and Dore

Before proceeding to the solution of equation (14) which was used in the body of the report, it is useful to present an alternate solution based on the method of Friedrich and Dore (ref. 2). Such a solution is applicable under somewhat more general circumstances than those of principal concern here and, hence, may prove valuable in other applications.

In the method of reference 2, it is necessary to assume that some technique is available for calculating the motion $\alpha(\gamma)$ from the entrance condition down to some point γ_P at which point $\alpha(\gamma_P)$ is at the peak of a cycle. Let this value of α be called α_P. Then an excellent approximate solution for the envelope curve of the α oscillation from this point forward is

$$\alpha_{max}(\gamma) = \alpha_P\left[\frac{M(\gamma_P)}{M(\gamma)}\right]^{\frac{1}{4}}\left(\frac{\cos\,\gamma_P - \cos\,\gamma_E}{\cos\,\gamma - \cos\,\gamma_E}\right)^{\frac{1}{2}} e^{K_1(\gamma_P - \gamma)} \tag{19}$$

where $M(\gamma)$ is given by equation (18). Equation (19) can be used to estimate the magnitude of the α motion under fairly wide variations of the atmospheric, aerodynamic, and inertial parameters. Further, using equation (19) and following the arguments of reference 2, one can go on to compute the detailed history of the angular oscillations, frequencies, etc. These results will not be presented here in full, since their derivation is well described in reference 2. Also, primarily because the expression for $M(\gamma)$ is very complicated, the full expressions are somewhat unwieldy. For the conditions of concern in this paper, however, it will be shown later that $M(\gamma)$ is given with good accuracy for all values of γ, except those very near γ_E, by

$$M(\gamma) = \frac{K_2}{\cos\,\gamma - \cos\,\gamma_E} \tag{20}$$

where

$$K_2 = -\frac{C_{m_\alpha}}{C_L}\frac{1}{\beta l}\left(\frac{l}{\sigma}\right)^2 \tag{21}$$

Since in equation (19) it must be assumed that $\gamma \leqq \gamma_P < \gamma_E$, the approximation (20) can be used, and equation (19) simplifies to

$$\alpha_{max}(\gamma) = \alpha_P\left(\frac{\cos\,\gamma_P - \cos\,\gamma_E}{\cos\,\gamma - \cos\,\gamma_E}\right)^{\frac{1}{4}} e^{K_1(\gamma_P - \gamma)} \tag{22}$$

Alternatively, the form of the dependency of α_{max} on atmospheric density can now be demonstrated explicitly by use of equation (10); thus

$$\alpha_{max}(\gamma) = \alpha_P\left[\frac{\rho(\gamma_P)}{\rho(\gamma)}\right]^{\frac{1}{4}} e^{K_1(\gamma_P - \gamma)} \tag{23}$$

Finally, under the assumption that the static flight path angle does not change radically from its initial value, applying the small-angle approximation to equation (10) permits equation (23) to be written as

$$\begin{aligned} \alpha_{max}(\gamma) &= \alpha_P\left[\frac{\rho(\gamma_P)}{\rho(\gamma)}\right]^{\frac{1}{4}} e^{\lambda_1[\rho(\gamma) - \rho(\gamma_P)]}, \qquad \gamma > 0 \\ &= \alpha_P\left[\frac{\rho(\gamma_P)}{\rho(\gamma)}\right]^{\frac{1}{4}} e^{\lambda_1[\rho(0) - \rho(\gamma_P)]} e^{\lambda_1[\rho(0) - \rho(\gamma)]}, \qquad \gamma < 0 \end{aligned} \tag{24}$$

with

$$\lambda_1 = \frac{AC_L K_1}{2\beta m\,\sin\,\gamma_E} \tag{25}$$

The first of equations (24) applies to the descending phase of the trajectory, the second, to the ascending phase. To the same order of accuracy as equation (22), the variation of oscillation frequency (in radians/sec) is found to be

$$\omega = \beta V_E \sqrt{K_2(\cos\,\gamma - \cos\,\gamma_E)}\; e^{\frac{\gamma - \gamma_E}{L/D}} \tag{26}$$

Alternatively, using equations (10) again, we may put equation (26) in the form

$$\omega = \sqrt{\frac{\beta C_L A K_2}{2m}}\, V(\gamma)\sqrt{\rho(\gamma)} \tag{27}$$

The variation of maximum angular velocity may be found approximately from equations (24) and (27) according to

$$\dot{\alpha}_{max}(\gamma) = \alpha_{max}(\gamma)\omega(\gamma) \tag{28}$$

It will be noted in equations (24), equation (27), and so, also in equation (28), that in effect the details of the particular trajectory under study have been eliminated. Hence, these expressions may be expected to yield at least qualitative estimates of α_{max}, $\dot{\alpha}_{max}$, and ω for other trajectories not too dissimilar from the one considered here. This point will be exploited later as a means of studying the effects of gravity on the motion.

Bessel Function Solutions

We now develop an approximate solution to equation (17) from whose form the main characteristics of the motion can be deduced with relative ease. Basically, the solution depends on finding an adequate approximation to $M(\gamma)$ over the entire range of γ. This can be done by considering the relative magnitudes of the terms in $M(\gamma)$ for some representative values of the parameters. Such a study was carried out (see appendix) and it was found that for all values of γ, $M(\gamma)$ can be approximated with excellent accuracy by

$$M(\gamma) = \frac{K_2}{\cos \gamma - \cos \gamma_E} + \frac{1 - \cos \gamma \cos \gamma_E}{4(\cos \gamma - \cos \gamma_E)^2} \tag{29}$$

Now since K_2 is of the order of 10^3, it is clear that the second term in equation (29) is trivial compared to the first for all values of γ except those very near γ_E. Since the Friedrich-Dore method may be applied only after the motion has completed at least one half-cycle, values of γ near γ_E are automatically excluded; hence the second term may be discarded and we have the approximation used previously in equations (20) and (22). Here, however, we seek another approximation which will include the first half-cycle. Such an approximation can be made if we require that γ_E, the entrance angle, be not too large. In this case, using the small-angle approximation, we have for $\gamma_E > \gamma > 0$

$$\cos \gamma - \cos \gamma_E = 2 \sin\left(\frac{\gamma_E - \gamma}{2}\right)\sin\left(\frac{\gamma_E + \gamma}{2}\right) \approx (\gamma_E - \gamma)\sin \gamma_E \tag{30}$$

Likewise, in the range $0 > \gamma > -\gamma_E$, the appropriate approximation is

$$\cos \gamma - \cos \gamma_E \approx (\gamma_E + \gamma)\sin \gamma_E \tag{31}$$

As for the second term in equation (29), since it is of significant magnitude only for values of γ near γ_E, a valid approximation to the numerator is required only for those values of γ. Hence,

$$1 - \cos \gamma \cos \gamma_E \approx 1 - \cos^2\gamma_E = \sin^2\gamma_E \tag{32}$$

Finally, therefore, using equations (30), (31), and (32), we have as approximations to $M(\gamma)$:

$$\left.\begin{aligned} &\gamma_E > \gamma > 0\,; \quad M(\gamma) = \frac{\kappa_2}{\gamma_E - \gamma} + \frac{1}{4(\gamma_E - \gamma)^2} \\ &0 > \gamma > -\gamma_E\,; \quad M(\gamma) = \frac{\kappa_2}{\gamma_E + \gamma} + \frac{1}{4(\gamma_E + \gamma)^2} \end{aligned}\right\} \tag{33}$$

where

$$\kappa_2 = \frac{K_2}{\sin \gamma_E}$$

The first of equations (33) is to be used for that part of the trajectory in which the vehicle descends through the atmosphere; the second, after the vehicle begins to ascend. The two phases will now be considered in turn.

Descent phase.- Letting the variable be $\eta = \gamma_E - \gamma$, and substituting the first of equations (33) in (17), we have for the equation of motion in the range $0 < \eta < \gamma_E$,

$$\bar{\alpha}''(\eta) + \left(\frac{\kappa_2}{\eta} + \frac{1}{4\eta^2}\right)\bar{\alpha}(\eta) = 0 \tag{34}$$

The solution to this equation is known (cf., ref. 6) and gives

$$\bar{\alpha}(\eta) = \sqrt{\eta}\left[aJ_0\left(2\sqrt{\kappa_2 \eta}\right) + bY_0\left(2\sqrt{\kappa_2 \eta}\right)\right] \tag{35}$$

where $J_0(z)$ and $Y_0(z)$ are, in Watson's notation, the zero-order Bessel functions of first and second kind, respectively. The exponent in equation (16) may be evaluated by straightforward integration, whereupon, reverting to the original variable, we have for the α motion

$$\alpha(\gamma) = e^{K_1(\gamma_E - \gamma)}\left[a_1 J_0\left(2\sqrt{\kappa_2(\gamma_E - \gamma)}\right) + a_2 Y_0\left(2\sqrt{\kappa_2(\gamma_E - \gamma)}\right)\right] \tag{36}$$

The constants a_1 and a_2 are of course to be evaluated by specifying the initial conditions. If we specify that the vehicle enter the atmosphere at initial angle of attack α_E and without angular velocity then the second term in equation (36) vanishes; hence the α motion in the descent phase is given by

$$\alpha(\gamma) = \alpha_E e^{K_1(\gamma_E - \gamma)} J_o\left(2\sqrt{\kappa_2(\gamma_E - \gamma)}\right) ; \qquad \gamma_E > \gamma > 0 \tag{37}$$

Ascent phase.- With the variable now $\eta = \gamma_E + \gamma$, substitution of the second of equations (33) in (17) shows that the equation of motion is of the same form as equation (34). Hence, the solution for $\overline{\alpha}(\eta)$ is again given by equation (35). After evaluating the exponent in equation (16) (with the limits of integration, $0 > \xi > \gamma$) we get

$$\alpha(\gamma) = e^{-K_1\gamma}\left[cJ_o\left(2\sqrt{\kappa_2(\gamma_E + \gamma)}\right) + dY_o\left(2\sqrt{\kappa_2(\gamma_E + \gamma)}\right)\right] \tag{38}$$

The constants of integration again may be evaluated from the initial conditions, which are, of course, the end values of the descent phase. From equation (37), with $\gamma = 0$, we have

$$\left.\begin{aligned}
\alpha(0) &= \alpha_E e^{K_1\gamma_E} J_o(\mu) \\
\alpha'(0) &= \alpha_E e^{K_1\gamma_E}\left[-K_1 J_o(\mu) + \sqrt{\frac{\kappa_2}{\gamma_E}}\, J_1(\mu)\right] \\
\mu &= 2\sqrt{\kappa_2\gamma_E}
\end{aligned}\right\} \tag{39}$$

Using equations (39) together with (38), we may evaluate the constants c and d, whereupon the α motion in the ascent phase is found to be

$$\alpha(\gamma) = \alpha_E e^{K_1(\gamma_E - \gamma)}\left\{\left[\frac{J_o(\mu)Y_1(\mu) + J_1(\mu)Y_o(\mu)}{J_o(\mu)Y_1(\mu) - J_1(\mu)Y_o(\mu)}\right]J_o\left(2\sqrt{\kappa_2(\gamma_E + \gamma)}\right) - 2\left[\frac{J_o(\mu)J_1(\mu)}{J_o(\mu)Y_1(\mu) - J_1(\mu)Y_o(\mu)}\right]Y_o\left(2\sqrt{\kappa_2(\gamma_E + \gamma)}\right)\right\} ; \qquad 0 > \gamma > -\gamma_E \tag{40}$$

Finally, it is clear that the same technique used to cast equations (24) to (28) solely in terms of atmospheric density may be used in the results (37) and (40) as well. Thus, we have as an alternate solution for the motion in the descent phase

$$\alpha(\gamma) = \alpha_E e^{\lambda_1\rho(\gamma)} J_o\left(2\sqrt{\lambda_2\rho(\gamma)}\right) ; \qquad \gamma_E > \gamma > 0 \tag{41}$$

Likewise, the solution in the ascent phase becomes

$$\alpha(\gamma) = \alpha_E e^{\lambda_1[\rho(0) - \rho(\gamma)]} e^{\lambda_1\rho(0)}\left\{\left[\frac{J_0(\mu)Y_1(\mu) + J_1(\mu)Y_0(\mu)}{J_0(\mu)Y_1(\mu) - J_1(\mu)Y_0(\mu)}\right]J_0\left(2\sqrt{\lambda_2\rho(\gamma)}\right) - 2\left[\frac{J_0(\mu)J_1(\mu)}{J_0(\mu)Y_1(\mu) - J_1(\mu)Y_0(\mu)}\right]Y_0\left(2\sqrt{\lambda_2\rho(\gamma)}\right)\right\}; \qquad 0 > \gamma > -\gamma_E \tag{42}$$

where

$$\left.\begin{aligned} \lambda_1 &= \frac{AC_L K_1}{2\beta m \sin \gamma_E} \\ \lambda_2 &= \frac{\kappa_2 C_L A}{2\beta m \sin \gamma_E} \\ \mu &= 2\sqrt{\lambda_2\rho(0)} \end{aligned}\right\} \tag{43}$$

DISCUSSION

Special Cases

It is instructive to study equations (37) and (40) under some special conditions, as the character of the motion then can be deduced almost by inspection. Thus, let us assume that the damping coefficient K_1 in equations (37) and (40) is small enough to be neglected, and let us examine in turn the three special cases: (1) $J_1(\mu) = 0$; (2) $J_0(\mu) = 0$; (3) $J_0(\mu)Y_1(\mu) + J_1(\mu)Y_0(\mu) = 0$.

Case (1); $J_1(\mu) = 0$.- Since in the descent phase $\alpha(\gamma)$ is dependent only on the Bessel function $J_0(2\sqrt{\kappa_2(\gamma_E - \gamma)})$, and since $J_1(\mu) = -J_0{}'(\mu)$, the condition $J_1(\mu) = 0$ is to be interpreted as meaning that α is at the peak of a cycle at the instant the static trajectory is at its lowest point. The α motion in the ascent phase then reduces to

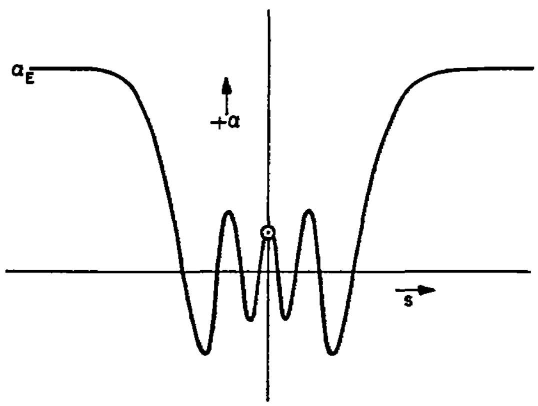
$$\alpha(\gamma) = \alpha_E J_0\left(2\sqrt{\kappa_2(\gamma_E + \gamma)}\right); \qquad 0 > \gamma > -\gamma_E \tag{44}$$

Hence, we find that the motion in the ascent phase is simply the mirror image of the motion in the descent phase; that is $\alpha(\gamma)$ is an even function about $\gamma = 0$ as shown in sketch (b).

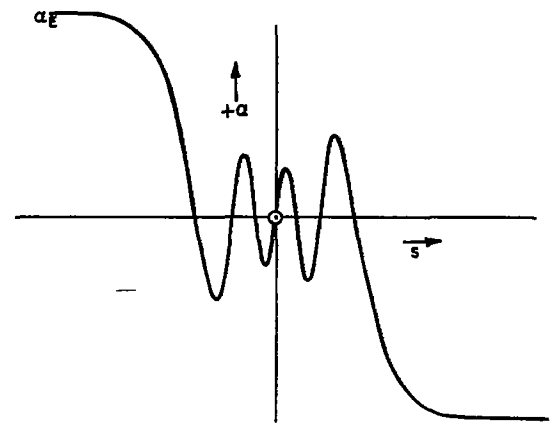
Case (2); $J_0(\mu) = 0$.- With $J_0(\mu) = 0$, α passes through zero at the lowest point of the static trajectory. Equation (40) reduces to

$$\alpha(\gamma) = -\alpha_E J_0\left(2\sqrt{\kappa_2(\gamma_E + \gamma)}\right); \qquad 0 > \gamma > -\gamma_E \tag{45}$$

Hence, the motion in the ascent phase is the negative mirror image of the descent phase; that is, $\alpha(\gamma)$ is an odd function about $\gamma = 0$ as shown in sketch (c).



Sketch (b)

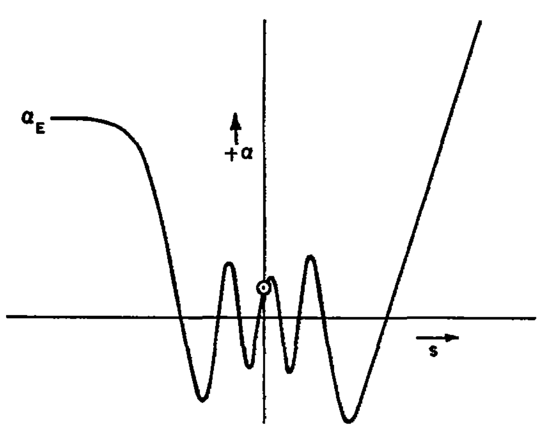


Sketch (c)

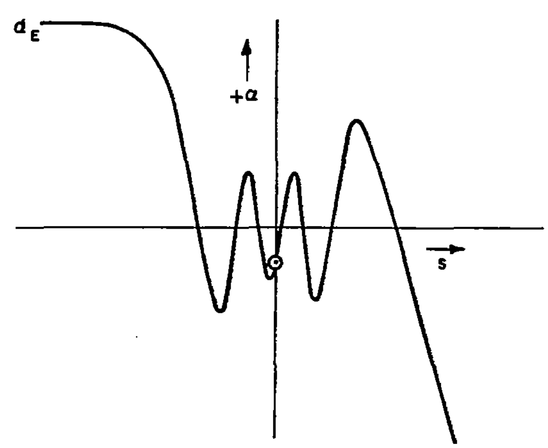
Case (3); $J_0(\mu)Y_1(\mu) + J_1(\mu)Y_0(\mu) = 0$.- Examination of this Bessel function combination for large values of the argument reveals that it is zero when α has completed either one-quarter or three-quarters of a cycle at $\gamma = 0$. Equation (40) reduces to

$$\alpha = \mp\alpha_E Y_0\left(2\sqrt{\kappa_2(\gamma_E + \gamma)}\right); \qquad 0 > \gamma > -\gamma_E \tag{46}$$

Note that, unlike the two previous cases, in this case the vehicle leaves the atmosphere with an angular velocity, as shown in sketches (d) and (e), since the Y_0 Bessel function approaches infinity as the argument approaches zero. Although $\alpha'(\gamma) \rightarrow \infty$ as $\gamma \rightarrow -\gamma_E$, nevertheless, the



Sketch (d)



Sketch (e)

angular velocity $\dot{\alpha}(t)$ is finite, since $\dot{\gamma}(t)\to 0$ as $\gamma\to -\gamma_E$. By the appropriate limiting process, the value of $\dot{\alpha}(t)$ acquired by the vehicle as it leaves the atmosphere may be calculated to be (for $K_1 \approx 0$ and case (3))

$$\left(\frac{d\alpha}{dt}\right)_{exit} = \mp \frac{1}{\pi} e^{\frac{-2\gamma_E}{L/D}} \alpha_E \beta V_E \sin \gamma_E \tag{47}$$

More generally, when K_1 is not neglected, and for an arbitrary value of $\alpha(0)$, the angular velocity at exit is

$$\left(\frac{d\alpha}{dt}\right)_{exit} = -\beta V_E \alpha_E \mu \sin \gamma_E J_0(\mu) J_1(\mu) e^{-2\gamma_E\left(\frac{1}{L/D} - K_1\right)} \tag{48}$$

Hence, for all but the special cases (1) and (2) we may say that the vehicle will tumble as it leaves the atmosphere, with a rate given by equation (48).

Oscillation History for Representative Vehicle

Let us now use the full equations (37) and (40) to determine the oscillation history of a vehicle representative of a type which might be suitable for flight at very high velocities and altitudes. The airframe selected consists of a low aspect ratio triangular wing mounted on a conical body, 50 feet in length. We assume it capable of developing an L/D of 6 at angle of attack of 0.1 radian. Calculations indicate that a representative value of the damping factor K_1 for such an airframe is of the order of -10; the static stability factor K_2 is of the order of 3×10^3 for a static margin of 3 percent. Let us also specify that the vehicle have an over-all range of 4000 nautical miles, so that, using results derivable from reference 4, we have that the vehicle enters the atmosphere on the skip phase of its trajectory with an initial velocity of 14,500 feet per second and with the flight path inclined initially at $\gamma_E = 12.2^\circ$.

In figure 1 is shown the history of the angle-of-attack oscillation together with the static skip trajectory. Also shown dotted is the history with the damping factor K_1 taken as zero. It will be noted that on the descent phase, even with zero damping, the motion subsides very quickly because of the powerful influence of increasing atmospheric density on the aerodynamic restoring moment. On the ascent phase, however, the aerodynamic damping is surprisingly effective in keeping the amplitudes small as the vehicle gains in altitude. Also, it should be noted that the value of μ for this example (eq. (43)) is such that α passes neither through zero nor a peak at the bottom of the skip; hence, the vehicle will tumble as it leaves the atmosphere. The tumbling rate does not appear to be very serious, however; for the present example, with $\alpha_E = 5^\circ$ and even with K_1 taken as zero, it is calculated to be of the order of one revolution every thirty minutes. With the damping factor included, this result is diminished seventyfold - to the order of one revolution every thirty-five hours.

General Form of Solution

If one examines the results of references 2, 3, and those of this paper (all dealing with the oscillatory motions developed by vehicles on entering the atmosphere), one finds a feature in common that is a departure from the familiar characteristics of conventional aircraft motions; namely, that the characteristic mode of oscillation is described by a Bessel function rather than a trigonometric function. Physically, the reason for this is clear since, as a vehicle descends through the atmosphere, the rapidly increasing atmospheric density causes the aerodynamic restoring moment to act in the manner of a stiffening spring. Thus, the oscillation must tighten both in amplitude and frequency, and this is precisely the behavior described by the Bessel function.

It already has been demonstrated (eqs. (41) and (42)) that the Bessel function solution for the oscillatory motion during the skip trajectory could be put in a form that is, in effect, independent of the type of trajectory. We wish now to show that this form will in fact recur for <u>any</u> trajectory, or any part of a trajectory, in which the direction of flight does not change rapidly with altitude. To this end, let us consider equation (6) again, and now change the independent variable to altitude y. The result is

$$\alpha''(y) + \alpha'(y)\left[\gamma'(y)\cot\gamma + \frac{1}{V}\frac{dV}{dy} - \frac{1}{V\sin\gamma}f_1(y)\right] + \frac{1}{V^2\sin^2\gamma}f_2(y)\alpha(y) = 0 \tag{49}$$

with

$$\left.\begin{aligned} f_1(y) &= \frac{\rho VA}{2m}\left[C_{L_\alpha} - \left(\frac{l}{\sigma}\right)^2\left(C_{m_q} + C_{m_{\dot\alpha}}\right)\right] \\ f_2(y) &= -\frac{\rho V^2 A}{2ml}\left(\frac{l}{\sigma}\right)^2 C_{m_\alpha} \end{aligned}\right\} \tag{50}$$

and where we have retained only the dominant term in $f_2(y)$. Now, in many cases (for example, on entering or leaving the atmosphere) γ will change only very slowly with altitude. Hence, let us assume

$$\gamma'(y) \approx 0\,, \qquad \sin\gamma \approx \text{constant} \tag{51}$$

Equation (49) becomes

$$\alpha''(y) + \alpha'(y)\left[\frac{1}{V}\frac{dV}{dy} + 2\beta\lambda_3\rho(y)\right] + \beta^2\lambda_2\rho(y)\alpha(y) = 0 \tag{52}$$

with

$$\left.\begin{aligned} \rho(y) &= \rho_o e^{-\beta y} \\ \lambda_3 &= \frac{A}{4\beta m \sin\gamma}\left[-C_{L_\alpha} + \left(\frac{l}{\sigma}\right)^2\left(C_{m_q} + C_{m_{\dot\alpha}}\right)\right] \\ \lambda_2 &= \frac{-A}{2\beta^2 ml \sin^2\gamma}\left(\frac{l}{\sigma}\right)^2 C_{m_\alpha} = \frac{AC_L K_2}{2\beta m \sin^2\gamma} \end{aligned}\right\} \tag{53}$$

Note that λ_3 and λ_2 have the same form as λ_1 and λ_2, respectively, in equation (43) except that now λ_3 does not contain the drag coefficient and γ is not necessarily the entrance flight path angle. Making the same transformation as before,

$$\alpha(y) = \bar{\alpha}(y) e^{-\frac{1}{2}\int_{y_1}^{y}\left[\frac{1}{V}\frac{dV}{d\xi} + 2\beta\lambda_3\rho(\xi)\right]d\xi} \tag{54}$$

we have

$$\bar{\alpha}''(y) + N(y)\bar{\alpha}(y) = 0 \tag{55}$$

with

$$N(y) = \beta^2\lambda_2\rho(y) - \frac{1}{2}\frac{d}{dy}\left[\frac{1}{V}\frac{dV}{dy} + 2\beta\lambda_3\rho(y)\right] - \frac{1}{4}\left[\frac{1}{V}\frac{dV}{dy} + 2\beta\lambda_3\rho(y)\right]^2 \tag{56}$$

Now since λ_2 is proportional to K_2 whereas the bracketed term is essentially proportional to K_1, it is easy to show (see appendix) that the contributions of the two terms containing λ_3 and V are small compared with that of the first term. Retaining only the first term in equation (56), we get for the differential equation

$$\bar{\alpha}''(y) + \beta^2\lambda_2\rho(y)\bar{\alpha}(y) = 0 \tag{57}$$

in which form the nature of the aerodynamic spring is clearly evident. The solution to equation (57) is

$$\bar{\alpha}(y) = c_1 J_o\left(2\sqrt{\lambda_2\rho(y)}\right) + c_2 Y_o\left(2\sqrt{\lambda_2\rho(y)}\right) \tag{58}$$

Consider now the exponential factor in equation (54). Recognizing that $\frac{1}{V}\frac{dV}{d\xi}$ can be written as $\frac{d}{d\xi}(\ln V)$, we see that this term is immediately integrable. The exponential term becomes

$$\sqrt{\frac{V(y_1)}{V(y)}}\, e^{-\beta\int_{y_1}^{y}\lambda_3\rho(\xi)d\xi} \tag{59}$$

Note that the damping factor λ_3 has been kept inside the integral, since by this means we extend the validity of the analysis to cases wherein λ_3 may be dependent on altitude. Then, combining equations (58) and (59), we have as a general form for the oscillatory motion

$$\alpha(y) = \frac{1}{\sqrt{V(y)}}\, e^{-\beta\int^{y}\lambda_3\rho(\xi)d\xi}\left[C_1J_0\left(2\sqrt{\lambda_2\rho(y)}\right) + C_2Y_0\left(2\sqrt{\lambda_2\rho(y)}\right)\right] \tag{60}$$

Equation (60) has the following virtues which make it particularly suited to studying the oscillatory history. First, it is clear that the effects on the motion of gravity, variations in the drag coefficient, and use of thrust are all automatically incorporated by using in equation (60) values of $V(y)$ derived from an accurate solution of the static trajectory equations. Second, one can account for the effects of variations with altitude of the damping parameter λ_3. Finally, equation (60) can be used to derive a simple criterion whose satisfaction ensures that the motion is convergent at any altitude. Thus, noting that the envelope of the oscillations varies according to

$$\alpha_{max}(y) = \frac{C(\gamma)e^{-\beta\int^{y}\lambda_3\rho(\xi)d\xi}}{[q(y)]^{\frac{1}{4}}} \tag{61}$$

we have that the motion is convergent during a descent provided that $\alpha'_{max}(y) > 0$. This is ensured if

$$-\beta\lambda_3(y)\rho(y) - \frac{1}{4}\frac{q'(y)}{q(y)} > 0 \tag{62}$$

Again, it should be noted that the relation (62) correctly includes gravity effects when these are included in $q(y)$. In this respect, equation (62) is a more precise statement of a similar result presented by Allen in reference 3, in which gravity effects are neglected.

Next, it is in order to show that the form of equation (60) does in fact reduce to the form derived previously (eqs. (41) and (42)) when gravity effects are not excessive. To see this, consider the first of the static trajectory equations (eqs. (4)). We have

$$\frac{1}{V}\frac{dV}{dy} = \frac{C_D A}{2m \sin \gamma} \rho(y) - \frac{g}{V^2}$$

$$= \frac{A\rho(y)}{2m \sin \gamma}\left[C_D - \frac{W}{A}\frac{\sin \gamma}{q}\right] \tag{63}$$

From the second of these forms, it is clear that the gravity term can be ignored over any part of a trajectory in which the inequality

$$\frac{q}{|\sin \gamma|} \gg \frac{W}{C_D A} \tag{64}$$

holds. Alternatively, as a more general expedient one may average the bracketed quantity in equation (63) in some sense and call it an effective constant drag coefficient, say C_{D_e}. In either of these circumstances, equation (63) can be integrated. Using the latter, we get

$$V = V(y_1)e^{-\frac{AC_{D_e}}{2\beta m \sin \gamma}\left[\rho(y) - \rho(y_1)\right]} \tag{65}$$

Combining this result with equation (60) for the case λ_3 = constant then gives

$$\alpha(y) = e^{\lambda_1 \rho(y)}\left[C_1 J_0\left(2\sqrt{\lambda_2 \rho(y)}\right) + C_2 Y_0\left(2\sqrt{\lambda_2 \rho(y)}\right)\right] \tag{66}$$

with

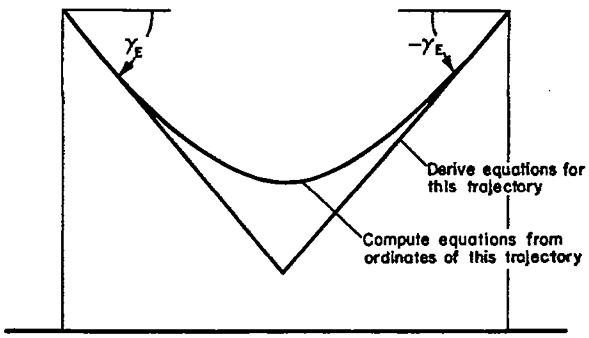
$$\lambda_1 = \frac{A}{4\beta m \sin \gamma}\left[C_{D_e} - C_{L_\alpha} + \left(\frac{l}{\sigma}\right)^2\left(C_{m_q} + C_{m_{\dot{\alpha}}}\right)\right]$$

Clearly, equations (41) and (42) are in accord with this form merely with the substitution of y for γ as the independent variable. Further, by making use of the properties of Bessel functions for large values of the argument, one can show that the amplitude and frequency of the oscillation vary essentially as

$$\left.\begin{aligned} \alpha_{max}(y) &= \frac{C(\gamma)e^{\lambda_1 \rho(y)}}{[\rho(y)]^{\frac{1}{4}}} \\ \omega(y) &= \sqrt{\frac{\beta A C_L K_2}{2m}}\, V(y)\sqrt{\rho(y)} \end{aligned}\right\} \tag{67}$$

in confirmation of equations (24) and (27). Finally, it should be emphasized that should the inequality (64) become an equality, C_{D_e} is to be put equal to zero in equation (66). This of course is also reflected in equation (60) by the fact that V becomes constant.

Arbitrary trajectories.- Although equations (60) and (66) are strictly applicable only while γ remains essentially constant, in many instances the fact that γ changes will be of no consequence, since, if the exponent is negative and the density increasing, the motion damps so rapidly that by the time γ has changed significantly α is only a small fraction of its initial value. This is the case for the ballistic trajectory (cf., ref. 3). On the other hand, should the sign of γ change over the trajectory, as it does for the skip path, or if greater accuracy is desired, then the trajectory can be broken into an appropriate number of straight-line segments (none having zero slope, however), over each of which equations (60) and (66) apply strictly. The parameters λ_1, λ_2, λ_3 and constants C_1, C_2 will then change from segment to segment, the first three being dependent on the value of $\sin \gamma$ (see eqs. (53) and (66)), the second two being determined so that the initial conditions at the start of a segment match the end conditions of the preceding segment. It is interesting to note that the results previously obtained as equations (41) and (42) for the skip trajectory can be reproduced by this procedure if, after deriving the equations corresponding to a static trajectory composed of two line segments inclined at γ_E and $-\gamma_E$, respectively, one then assigns values to the ordinate y corresponding to those of the actual trajectory. Sketch (f) illustrates the procedure, which obviously can be extended to apply to arbitrary trajectories.



Sketch (f)

Effect of gravity.- The obvious advantage of equations (60) and (66) is that they may be used with any static trajectory whenever the altitude and velocity histories are given. As an immediate consequence, we may now evaluate the errors in our former results (eqs. (37) and (40)) caused by having neglected the gravity terms in the solution of equations (4) for the static skip trajectory.

In order to carry out such a study, the static trajectory equations (4) with gravity terms included were solved numerically for the same conditions as were used for figure 1.[1] Having the new altitude and velocity histories, we could then use equations (61) and (67) to estimate the variations in α_{max} and ω corresponding to this trajectory. Results for the static trajectory and velocity histories with gravity included and omitted are shown in figures 2(a) and 2(b), and the corresponding

[1]The term $mV^2 \cos \gamma_s/(R + y)$ in the second of equations (4) was neglected in these calculations in order to isolate the effects of gravity. For velocities of the magnitude considered here, its effect on the trajectory is small compared with that of gravity, and in the direction of reducing the differences between the results with gravity terms included and omitted.

variations of α_{max} and ω in figures 2(c) and 2(d). It will be noted in figure 2(a) that because of the initial gain in flight speed (fig. 2(b)), the true static trajectory descends sooner into the atmosphere and also deeper, by about 15,000 feet. The effects of these changes on the oscillatory angle-of-attack envelope (fig. 2(c)) are evidently not excessive; however, the frequencies are significantly increased (fig. 2(d)). This is a reflection of the fact that whereas the envelope curve varies as the fourth root of density, the frequency varies only as the square root.

We should note that the above effects of gravity diminish as the trajectory becomes steeper. To illustrate this, a set of curves paralleling figure 2 has been prepared for the case of a skip rocket having an L/D of 3 rather than 6. In order to achieve the same 4000 nautical mile range, the entrance angle in this event must be increased from 12.2° to 18.6° and the entrance velocity from 14,500 to 18,200 ft/sec. The results are shown in figures 3(a) to 3(d), and it is clear that here the influence of gravity is not significant either as it affects the static trajectory (figs. 3(a) and 3(b)) or the oscillatory quantities (figs. 3(c) and 3(d)). On the other hand, however, since the accelerative loads become very large during steep skips, it would appear that a shallow trajectory of the type shown in figure 2(a) might be a more likely prospect for practical consideration than the one shown in figure 3(a). For such shallow flight paths, in order to obtain a static trajectory of sufficient accuracy, the gravity terms probably ought not be discarded in equations (4). Once having obtained the static trajectory, however (by whatever means), equation (60) or (66) can always be used to compute the oscillatory motion.

CONCLUDING REMARKS

An analysis has been carried out of the oscillatory motions of a vehicle which traverses a path through the atmosphere at high speed. The main results may be summarized briefly as follows: For any part of a static trajectory in which the flight path angle does not vary greatly with altitude, the oscillatory angle of attack and circular frequency will have the form

$$\alpha(y) = \frac{e^{-\beta \int^{y} \lambda_3 \rho(\xi) d\xi}}{\sqrt{V(y)}} \left[C_1 J_o\left(2\sqrt{\lambda_2 \rho(y)}\right) + C_2 Y_o\left(2\sqrt{\lambda_2 \rho(y)}\right)\right]$$

$$\omega(y) = KV(y)\sqrt{\rho(y)}$$

where $\rho(y)$ and $V(y)$ are, respectively, the atmospheric density and flight speed as functions of altitude. Since a trajectory can always be decomposed into straight-line segments, over each of which the above equations apply, it is possible to say that given the altitude and velocity history of a static trajectory, the corresponding oscillatory motion can always be computed by the use of these equations.

Ames Aeronautical Laboratory
National Advisory Committee for Aeronautics
Moffett Field, Calif., Feb. 25, 1958

APPENDIX

SIMPLIFIED EXPRESSION FOR $M(\gamma)$

The purpose in this appendix is to demonstrate that, for representative values of the parameters, the coefficient $M(\gamma)$ can in fact be simplified to the form given in the text as equation (29).

Written in full, $M(\gamma)$ is expressible as

$$M(\gamma) = \frac{1}{(\cos\gamma - \cos\gamma_E)}\left[\sin\gamma\left(\frac{C_{L_\alpha}}{C_L} + K_1\right) + \frac{1}{4}\cos\gamma + K_2\right] + \frac{1 - \cos\gamma\cos\gamma_E}{4(\cos\gamma - \cos\gamma_E)^2} - \left[\frac{C_{L_\alpha}}{C_L}\frac{1}{L/D} + \frac{C_{m_q}C_{L_\alpha}}{C_L^2}\left(\frac{l}{\sigma}\right)^2 + K_1^2\right] \tag{A1}$$

where

$$K_1 = \frac{1}{2}\left[\frac{1}{L/D} - \frac{C_{L_\alpha}}{C_L} + \left(\frac{C_{m_q} + C_{m_{\dot\alpha}}}{C_L}\right)\left(\frac{l}{\sigma}\right)^2\right]$$

$$K_2 = -\frac{C_{m_\alpha}}{C_L}\frac{1}{\beta l}\left(\frac{l}{\sigma}\right)^2$$

Hence, (comparing equation (A1) with equation (29)) we must demonstrate that the quantity within the brackets of the first term in equation (A1) is essentially K_2 and that the third term is negligible compared with the other two.

Consider now an airplane configuration which might be selected for flight at very high altitudes and velocities. Let it have a low aspect ratio triangular wing mounted on a conical body. A generous L/D for such a configuration is 6. As further specifications, let the vehicle have an over-all length of 50 feet, a trim angle of attack of 5.73° and (to prejudice the case against the argument) a static margin of only 1 percent, measured in body lengths. Then

$$\left.\begin{aligned} \frac{L}{D} &= 6 \\ \frac{C_{L_\alpha}}{C_L} &= 10 \\ \frac{C_{m_\alpha}}{C_L} &= -\frac{1}{10} \end{aligned}\right\} \tag{A2}$$

and, using linearized potential theory for triangular wings having supersonic edges,

$$\left.\begin{aligned}\frac{C_{m_q} + C_{m_{\dot{\alpha}}}}{C_L} &\approx -\frac{1}{2} \\ \frac{C_{m_q} C_{L_\alpha}}{{C_L}^2} &\approx -6\end{aligned}\right\} \tag{A3}$$

Assuming the body to be uniformly solid, we find the moment of inertia about the center of gravity is essentially

$$I \approx \frac{1}{22} m l^2 \tag{A4}$$

so that

$$\left(\frac{l}{\sigma}\right)^2 \approx 22 \tag{A5}$$

Hence, with

$$\frac{1}{\beta l} = \frac{22,000}{50} = 440$$

$$\left.\begin{aligned}K_2 &= 968 \\ K_1 &= -10.4 \\ \left[\frac{C_{L_\alpha}}{C_L}\frac{1}{L/D} + \frac{C_{m_q} C_{L_\alpha}}{{C_L}^2}\left(\frac{l}{\sigma}\right)^2 + {K_1}^2\right] &= -22.1\end{aligned}\right\} \tag{A6}$$

Obviously, then, even with an extremely small static margin, the influence of K_2 is overriding in the numerator of the first term of equation (A1). Further, since the smallest the first term can become is $K_2/(1 - \cos\gamma_E)$, the third term is negligible compared with the first for all values of γ. Therefore, $M(\gamma)$ is given with negligible loss in accuracy by

$$M(\gamma) = \frac{K_2}{\cos\gamma - \cos\gamma_E} + \frac{1 - \cos\gamma\cos\gamma_E}{4(\cos\gamma - \cos\gamma_E)^2} \tag{A7}$$

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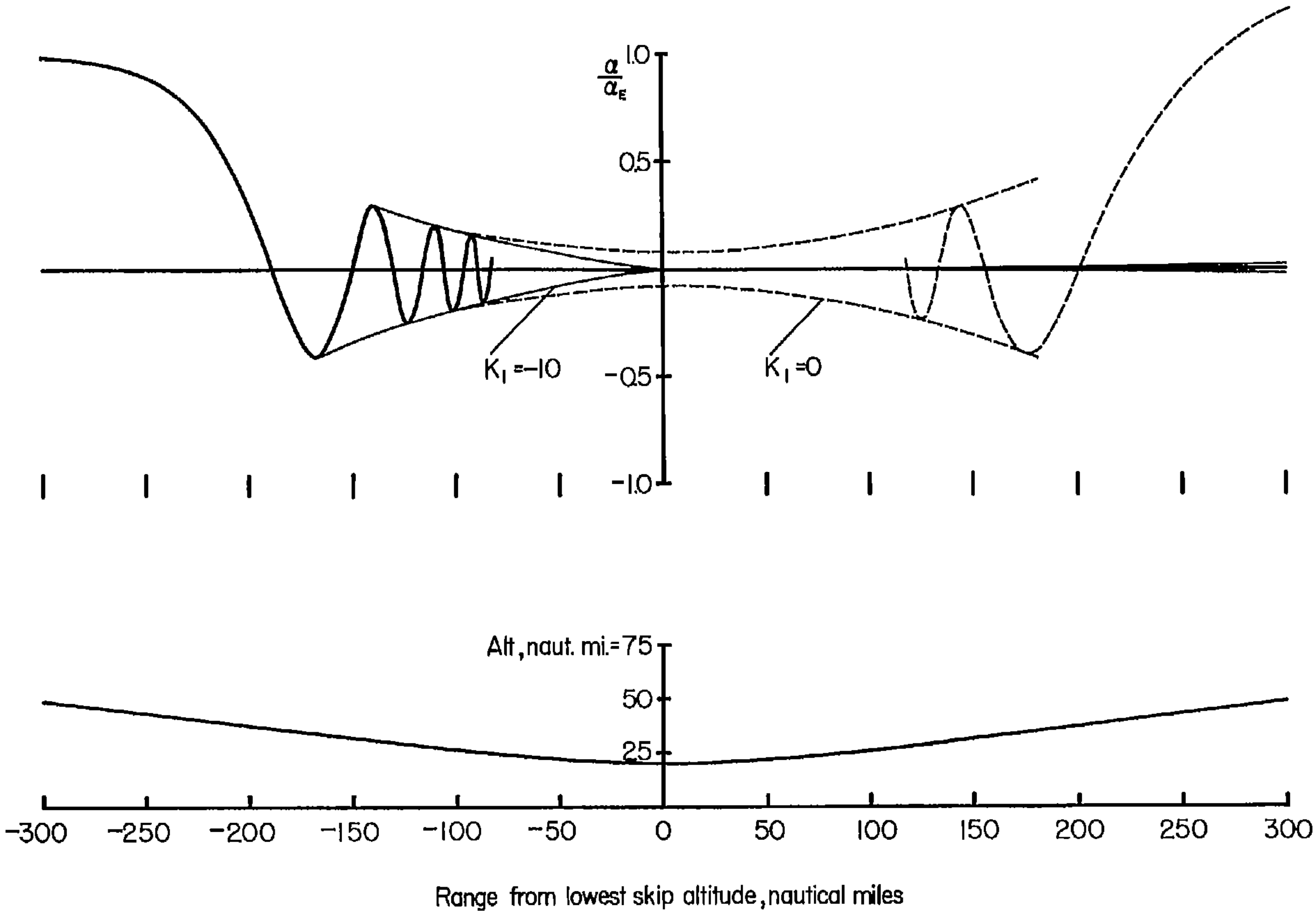
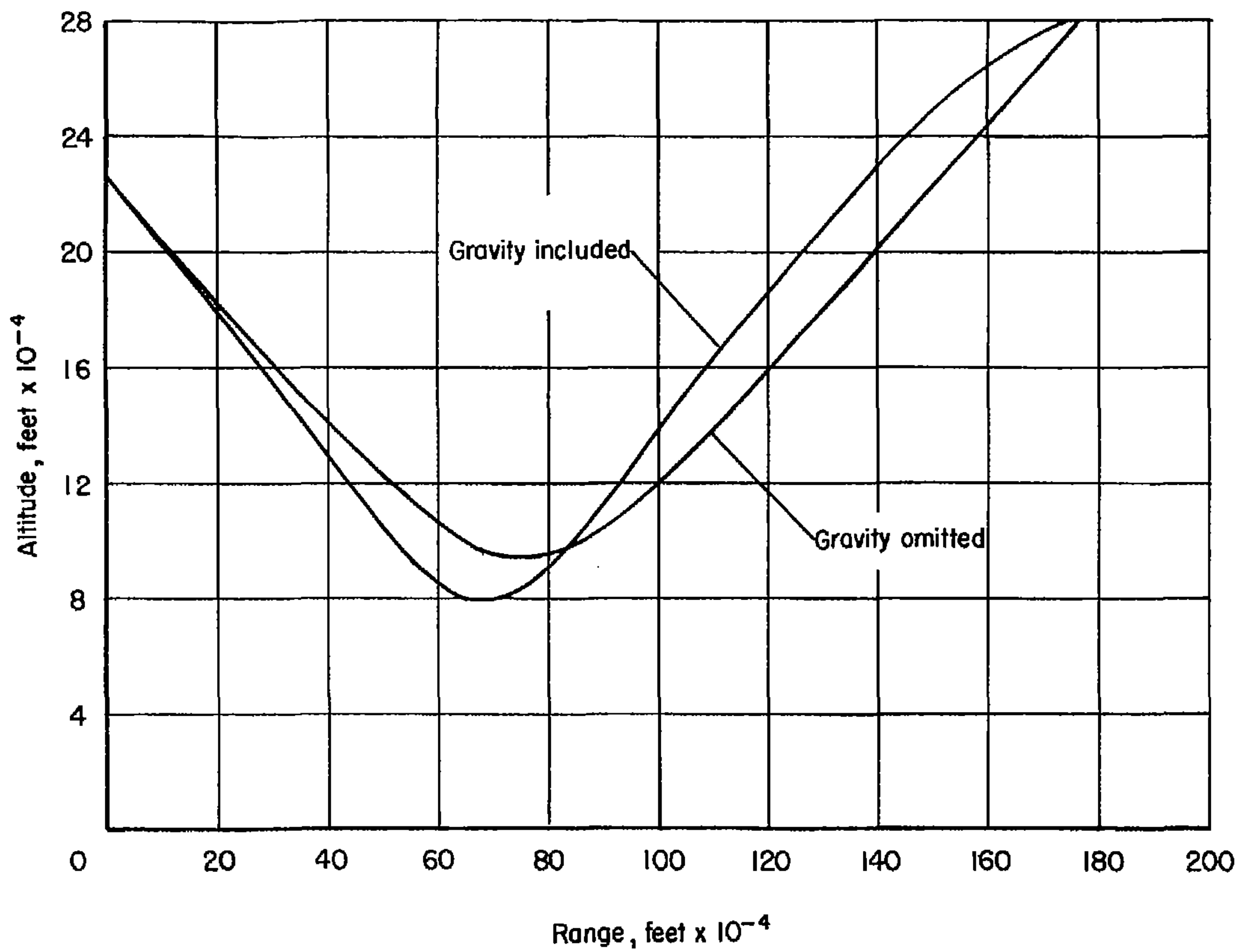
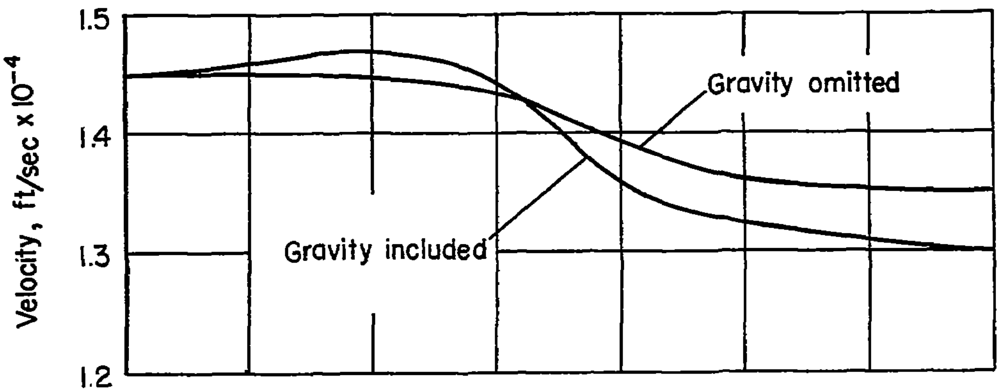


Figure 1.- Angle-of-attack oscillations and static trajectory of a skip rocket having an L/D of 6 and an over-all range of 4000 nautical miles.

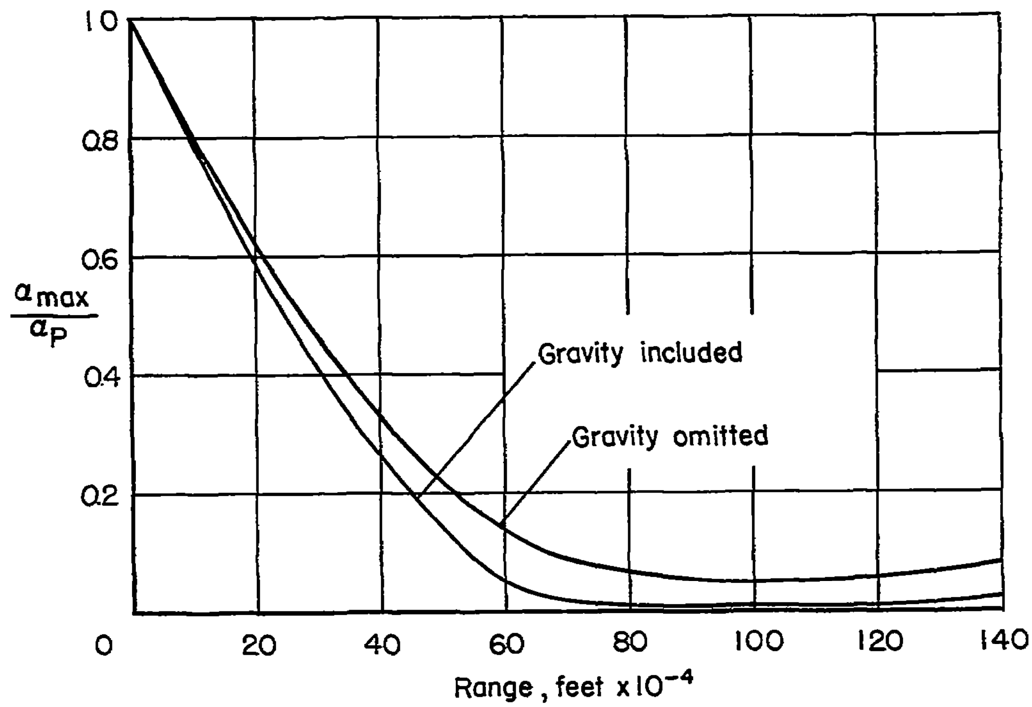


(a) Static trajectory.

Figure 2.- Effect of gravity on trajectory and oscillatory history of a skip rocket; L/D = 6; over-all range = 4000 nautical miles.

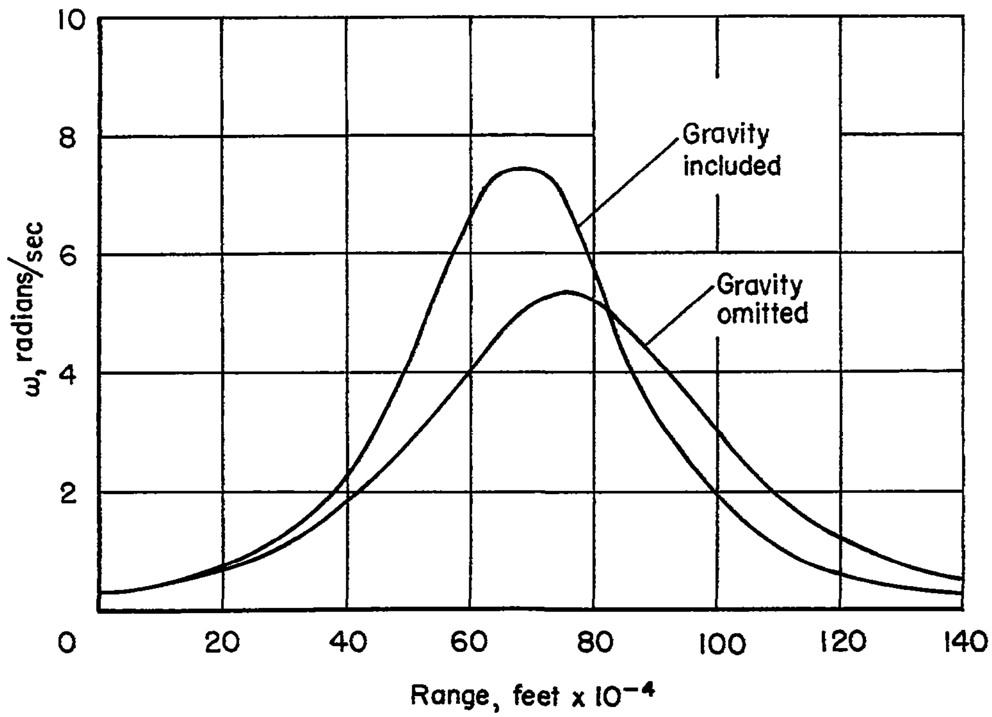


(b) Flight speed.



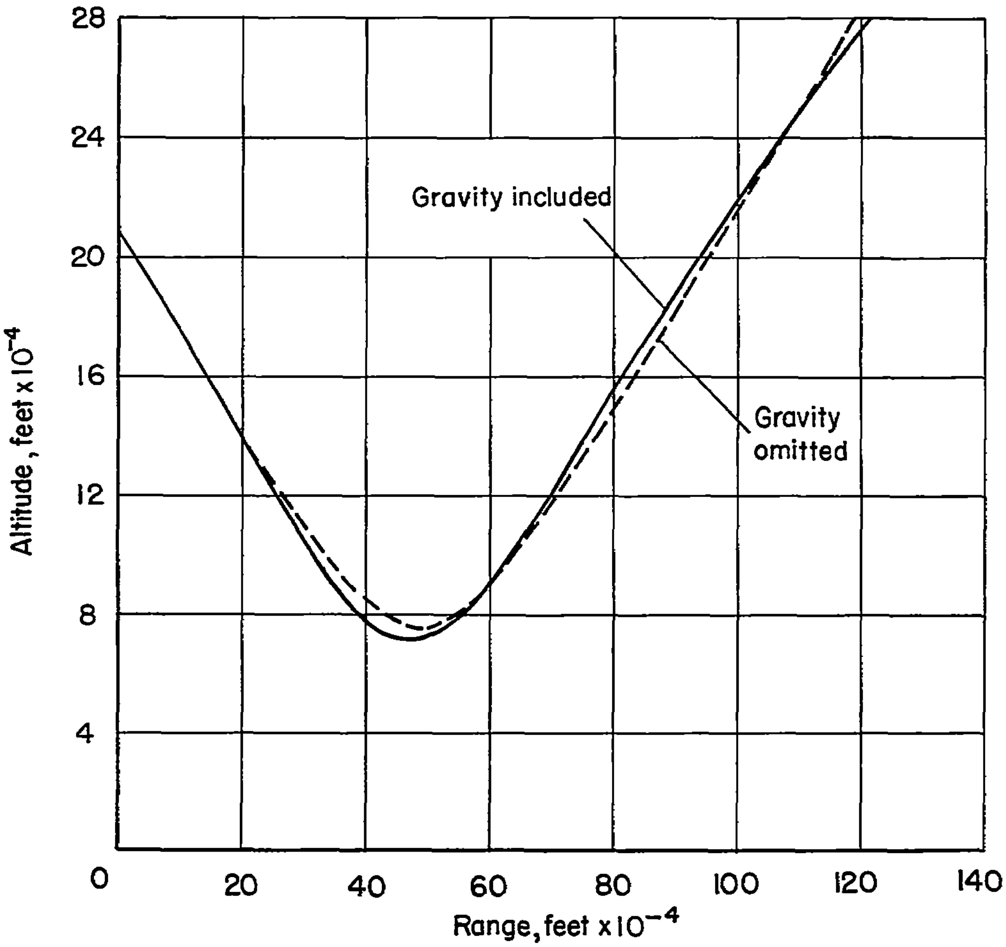
(c) Envelope of angle-of-attack oscillations.

Figure 2.- Continued.



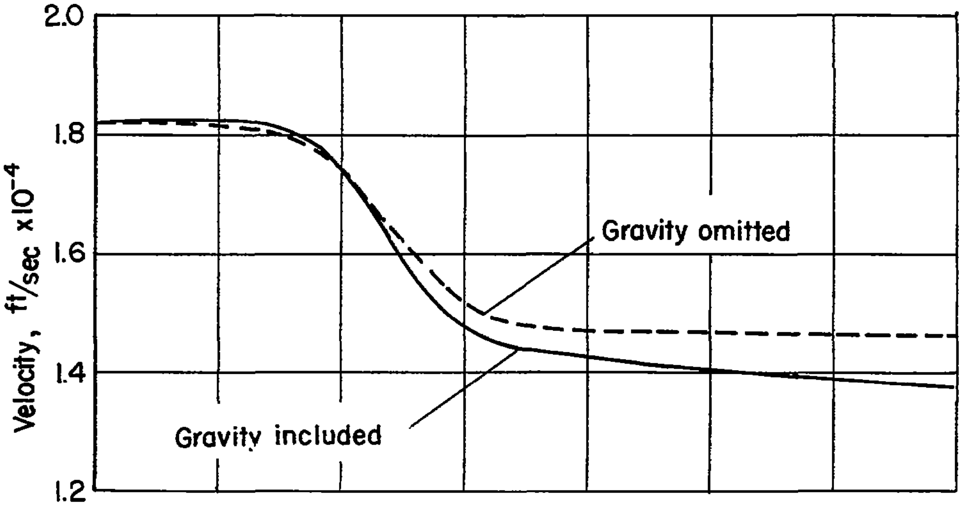
(d) Oscillation frequency.

Figure 2.- Concluded.

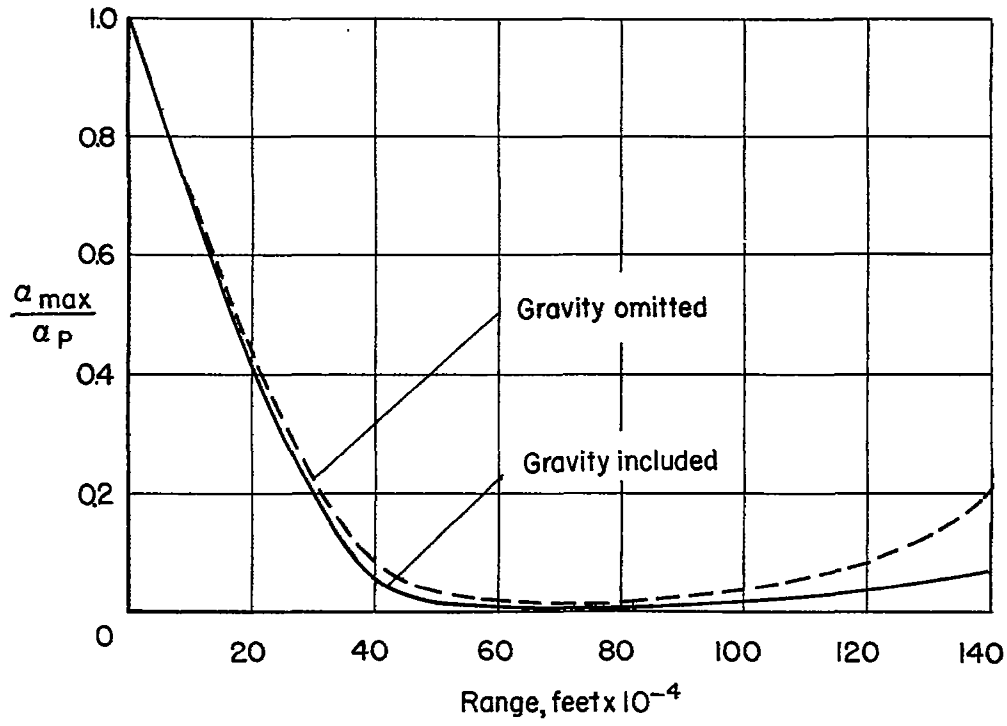


(a) Static trajectory.

Figure 3.- Effect of gravity on trajectory and oscillatory history of a skip rocket; L/D = 3; over-all range = 4000 nautical miles.

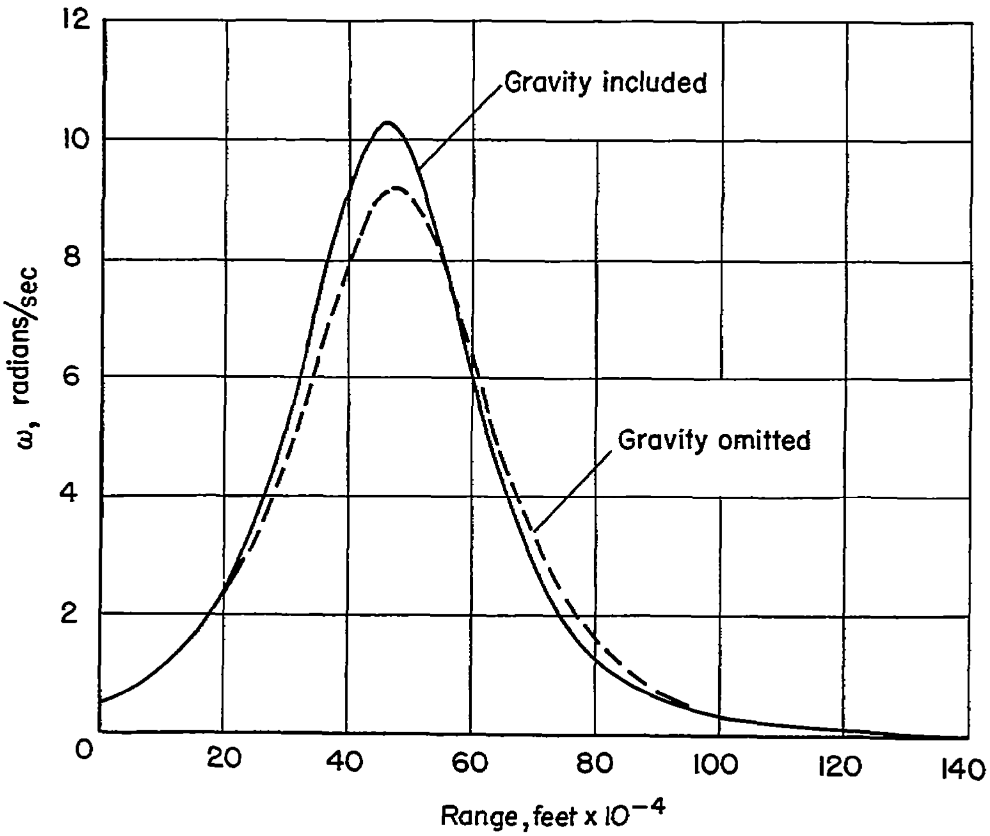


(b) Flight speed.



(c) Envelope of angle-of-attack oscillations.

Figure 3.- Continued.



(d) Oscillation frequency.

Figure 3.- Concluded.